Events and Probabilities

- A process is random if it is known that when it takes place, one outcome from a possible set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.
- A sample space is the set of all possible outcomes of a random process or experiment.
- An event is a subset of a sample space.
- If S is a finite sample space **in which all outcomes are equally likely**, and E is an event in S, then the probability of E, denoted P(E) is

$$P(E) = \frac{\text{the number of outcomes in E}}{\text{the total number of outcomes in S}}$$

Probability Properties

Let S be a sample space and A, B events in S

- $P(\emptyset) = 0$ and $P(S) = 1$
- $0 \le P(A), P(B) \le 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A^c) = 1 - P(A)$

Corollary: If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$

P32: Jar contains 4 red and 6 blue marbles. 2 are picked without replacement. What is the probability that the second marble is red?

Event	Description	N(Event)	P(Event)
S	sample space		
R_1	1[st] marble is red		
$B_1=R_1^C$	1st marble is blue (not red)		
R_2	2[nd] marble is red		
$R_1 \cap R_2$	Red - Red		
$B_1 \cap R_2$	Blue - Red		

P33: In the same situation, what is the probability that at least one of them is red?

Expected Value

> If the possible outcomes of an experiment, or random process, are real numbers a1 to an, which occur with probabilities p1 to pn respectively, then the expected value of the process is $\sum_{k=1}^{n} a_k p_k$

P34: A raffle for a charity sells 10,000 tickets @ $10 each. There are:

- 1 prize of $1,000
- 10 second prizes of $100
- 100 third prizes of $20

P35: Jar from P32-P33 contains 4 red and 6 blue marbles. 2 are picked without replacement. What is the expected value of the number of red marbles?

Conditional Probability

Next set of problems:

A family has two cisgendered children, each of whom is equally likely to be a boy or girl. We are going to look at different situations about additional information you may have about these children and how useful this information is in figuring out the gender configuration of this family.The situations (problems) are independent of each other.

P36: You meet the first born child who is a boy. What is the probability that the second born child is a girl?

P37: Suppose that you find out that one of the children is a boy. What is the probability that the other child is a girl?

> Let A and B be events in a sample space S. If P(A)≠0, then the conditional probability of B given A, denoted P(B|A), is P(B|A) = $\frac{P(A \cap B)}{P(A)}$

P38: **Different problem:** roll 2 dice: one red and one blue. What is the probability that the sum of the rolls is 8 given that both dice are odd?

Event	Description	Elements	N()	P()
S	sample space	$\{1,2,3,4,5,6\}^2$	36	
A	both dice are odd			
B	sum of rolls is 8			
$A \cap B$				

P39: Same question as P37, but what is the probability that the other child is a boy? i.e. Suppose that you find out that one of the children is a boy. What is the probability that the other child is a boy?

Event	Description	Elements	N()	P()
S	sample space	{BB,BG,GB,GG}	4	
A	one of the children is a boy	{BB,BG,GB}	3	3/4
B	at least one of the children is a girl	{GG,BG,GB}	3	3/4
B^c	none of the children are girls			
$A \cap B^c$	both children are boys			

> Theorem: $P(B^c|A) = 1 - P(B|A)$

Baye's Theorem

- Idea: determine the probability of an even based on knowledge of relationships between this event and other factors.
- e.g. if cancer is related to age (or other factors such as exposure to known carcinogens), then how can a person's age or history be used to assess the probability that they have cancer, compared to assessing that probability without knowing the person's age or history.

P40: Consider a medical test for a disease which affects 5 people out of a 1000. This test has a false positive rate of 3% and a false negative rate of 1%. The public wants to know how reliable the test is,
i.e: what is the probability that:

- A person who tests positive has the disease?
- A person who tests negative does not have the disease?

Event	Description	P(Event)
D	having disease	
H	not having disease (healthy)	
T^+	testing positive (i.e. for having disease)	
T^-	testing negative (i.e. not having disease)	
$T^+\|H$	False positive: test is positive for a healthy patient	
$T^+\|D$	True positive = 1 – $P(T^-\|D)$	
$T^-\|D$	False negative: test is negative for a sick patient	
$T^-\|H$	True negative = 1 - $P(T^+\|H)$	
$D\|T^+$	Person who tests positive has the disease	
$H\|T^-$	Person who tests negative does not have the disease	

Baye's Theorem: Given two events A and B s.t. P(B)≠0,
P(A|B) = P(B|A).P(A)/P(B)

Generalised Baye's: Suppose that a sample space S is a union of mutually disjoint events A1, A2, …, An, and suppose B is an event in S with P(B)≠0.

If k is an integer with 1≤k≤n, then P(Ak|B) = $\frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$

- Explanation:
  - the denominator is P(B) = $\bigcup_{i=1}^{n} P(B \cap A_i)$
  - = $\sum_{i=1}^{n} P(B|A_i)P(A_i)$

Independent Events

P41: **Back to P36 and P37**:

- 36: what is the probability that the second born child is a girl, given that the first born is a boy?
- 37: what is the probability that the family has a girl given that they have at least one boy

	Event	Description	Elements	N()	P()
	S	sample space	{BB,BG,GB,GG}	4	1
P36	A	First child is a boy	{BB, BG }	2	½
	B	Second child is a girl	{BG, GG}	2	½
	$A \cap B$		{BG}	1	¼
P37	A	one of the children is a boy	{BB,BG,GB}	3	¾
	B	at least one of the children is a girl	{GG,BG,GB}	3	¾
	$A \cap B$	one of the children is a boy and the other is a girl	{BG, GB}	2	½

- In P36, $P(B|A) = ½$ in P37, $P(B|A) = 2/3$

Let A, B and C be events in a sample space S.

A and B are independent iff $P(A \cap B) = P(A).P(B)$

P42: Jar from P32 contains 4 red and 6 blue marbles. 2 are picked without replacement. What is the probability that the second marble is red given that the first one is blue?

Event	Description	N(Event)	P(Event)
S	sample space	$10 \times 9 = 90$	
R_1	1^{st} marble is red		2/5
B_1	1st marble is blue		3/5
R_2	2^{nd} marble is red		2/5
$B_1 \cap R_2$	Blue - Red	6×4	$6 \times 4/90 = 4/15$

Independence and complement

P43: When rolling a die twice:
- what is the probability that the 2nd roll is a 5 given that the 1st is a 6?
- what is the probability that the 2nd roll is **not** a 5 given that the 1st is a 6?

Event	Description	Elements	N()	P()
S	sample space	$\{1,2,3,4,5,6\}^2$	36	
A	First roll is a 6			
B	Second roll is a 5			
B^c	Second roll is a not 5			
$A\cap B$				
$A\cap B^c$				

Theorem: If A and B are independent events, then so are A and B^c

- Question: If A and B are events with non-zero probabilities s.t. $B\subseteq A^c$, are A and B independent?

More generally: if $P(A)\neq 0$ and $P(B)\ )\neq 0$ but $A\cap B=\emptyset$, then the events A and B are not independent

Pairwise and Mutual Independence

P44: When rolling a die 3 times, what is the probability of rolling three 6s?

Event	Description	Elements	N()	P()
S	sample space	$\{1,2,3,4,5,6\}^3$	216	
A_1	1^{st} roll is a 6			
A_2	2^{nd} roll is a 6			
A_3	3^{rd} roll is a 6			
$A_1 \cap A_2 \cap A_3$	6 in all 3 rolls			

A, B, and C are pairwise independent iff they satisfy all the following conditions:

- $P(A \cap B) = P(A).P(B)$
- $P(B \cap C) = P(B).P(C)$
- $P(C \cap A) = P(C).P(A)$

A, B, and C are mutually independent iff they are pairwise independent and $P(A \cap B \cap C) = P(A).P(B).P(C)$

Generally: events A_1 to A_n are mutually independent iff the probability of the intersection of any subset of the events is the product of the probabilities of the events in the subset.

- Question: can you draw a diagram of three events which are all pairwise independent but not mutually independent?

P45: Loaded coin. A coin is loaded so that P(head)=0.6 and P(tail)=0.4. The coin is tossed 10 times. What is the probability of obtaining 7 heads?

IF TIME PERMITS

P46: Two jars of red and blue marbles: jar 1 has 3 blue and 4 red, and jar 2 has 5 blue and 3 red. A marble is selected by choosing one of the jars at random and picking a marble at random from the jar. If the chosen marble is blue, what is the probability that it came for the first jar?